

RECURSIVE STATE ESTIMATION FOR NONCAUSAL DISCRETE-TIME DESCRIPTOR SYSTEMS UNDER UNCERTAINTIES

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ABSTRACT. This paper describes a method for the online state estimation of systems described by a general class of linear noncausal time-varying difference descriptor equations subject to uncertainties. The method is based on the notions of a *linear minimax estimation* and an *index of causality* introduced here for singular difference equations. The *online minimax estimator* is derived by the application of the dynamical programming and Moore's pseudoinverse theory to the minimax estimation problem. It coincides with Kalman's filter for regular systems. A numerical example of the state estimation for 2D noncasual descriptor system is presented.

Keywords Kalman filtering, online state observer, guaranteed estimation, descriptor systems, singular systems, DAEs.

1. INTRODUCTION

There is a number of physical and engineering objects most naturally modelled as systems of differential and algebraic equations (DAEs) or descriptor systems: microwave circuits [1], flexible-link planar parallel platforms [2] and image recognition problems (noncasual image modelling) [4]. DAEs arise in economics [5]. Also non-linear differential-algebraic systems are studied with help of DAEs via linearization: a batch chemical reactor model [3].

On the other hand there are many papers devoted to the mathematical processing of data obtained from the measuring device during an experiment. In particular, a problem of the observer design for discrete-time descriptor systems was studied in the [5]-[8], the guaranteed state estimation for a linear dynamical systems was investigated in the [9]. In the [6] the authors derive a so-called "3-block" form for the optimal filter and a corresponding 3-block Riccati equation using a maximum likelihood approach. A filter is obtained for a general class of time-varying descriptor models. The measurements are supposed to contain a noise with Gauss'es distribution. The obtained recursion is stated in terms of the 3-block matrix pseudoinverse.

In the [7] the filter recursion is represented in terms of a deterministic data fitting problem solution. The authors introduce an explicit form of the 3-block matrix pseudoinverse for a descriptor system with a special structure: so their filter coincides with obtained in the [6].

In this paper we study an observer design problem for a general class of linear noncasual time-varying descriptor models with no restrictions to a system structure. Suppose we are given an exact mathematical model of some real process and the vector x_k describes the system output at the moment k in the corresponding state space of the system. Also the successive measurements $y_0 \dots y_k \dots$ of the system

output x_k are supposed to be available with the noise $g_0 \dots g_k \dots$ of an uncertain nature¹. Further assume that the system input f_k , start point q and noise g_k are arbitrary elements of the given set G . The aim of this paper is to design a minimax observer $k \mapsto \hat{x}_k$ that gives an online guaranteed estimation of the output x_k on the basis of measurements y_k and the structure of G . In [8] minimax estimations were derived from the 2-point boundary value problem with the conditions at $i = 0$ (start point) and $i = k$ (end point). Hence a recalculation of the whole history $\hat{x}_0 \dots \hat{x}_k$ is required if the moment k changes. Here we derive the observer $(k, y_k) \mapsto \hat{x}_k$ by applying dynamical programming methods to the minimax estimation problem similar to posed in the [8]. We construct a map \hat{x} that takes (k, y_k) to \hat{x}_k making it possible to assign a unique sequence of estimations $\hat{x}_0 \dots \hat{x}_k \dots$ to the given sequence of observations $y_0 \dots y_k \dots$ in the real time. A resulting filter recursion is stated in terms of the pseudoinverse of positive semi-defined $n \times n$ -matrices.

2. MINIMAX ESTIMATION PROBLEM

Assume that $x_k \in \mathbb{R}^n$ is described by the equation

$$(1) \quad F_{k+1}x_{k+1} - C_k x_k = f_k, k = 0, 1, \dots,$$

with the initial condition

$$(2) \quad F_0 x_0 = q,$$

and y_k is given by

$$(3) \quad y_k = H_k x_k + g_k, k = 0, 1, \dots,$$

where F_k, C_k are $m \times n$ -matrices, H_k is $p \times n$ -matrix. Since we deal with a descriptor system we see that for any k there is a set of vectors $x_1^0 \dots x_k^0$ satisfying (1) while $f_i = 0, q = 0$. Thus the undefined inner influence caused by $x_1^0 \dots x_k^0$ is possible to appear in the systems output. Also we suppose the initial condition q , input $\{f_k\}$ and noise $\{g_k\}$ to be unknown elements of the given set²

$$(4) \quad G(q, \{f_k\}, \{g_k\}) = (Sq, q) + \sum_0^\infty (S_k f_k, f_k) + (R_k g_k, g_k) \leq 1$$

where S, S_k, R_k are some symmetric positive-defined weight matrices with the appropriate dimensions. The trick is to fix any N -partial sum of (4) so that $(q, \{f_k\}, \{g_k\})$ belongs to

$$(5) \quad \mathcal{G}^N := \{(q, \{f_k\}, \{g_k\}) : (Sq, q) + \sum_{k=0}^{N-1} (S_k f_k, f_k) + \sum_{k=0}^N (R_k g_k, g_k) \leq 1\}$$

Then we derive the estimation $\hat{x}_N = v(N, y_N, \hat{x}_{N-1})$ considering a minimax estimation problem for \mathcal{G}^N . Lets denote by \mathcal{N} a set of all $(\{x_k\}, q, \{f_k\})$ such that (1) is held. The set \mathcal{G}_y^N is said to be *a-posteriori set*, where

$$(6) \quad \mathcal{G}_y^N := \{(\{x_k\}) : (\{x_k\}, q, \{f_k\}) \in \mathcal{N}, (q, \{f_k\}, \{y_k - H_k x_k\}) \in \mathcal{G}^N\}$$

¹For instance we do not have a-priori information about its distribution.

²Here and after (\cdot, \cdot) denotes an inner product in an appropriate euclidean space, $\|x\| = (x, x)^{\frac{1}{2}}$.

It follows from the definition that \mathcal{G}_y^N consists of all possible $\{x_k\}$ causing an appearance of given $\{y_k\}$ while $(q, \{f_k\}, \{g_k\})$ runs through \mathcal{G}^N . Thus, it's naturally to look for x_N estimation **only** among the elements of $P_N(\mathcal{G}_y^N)$, where P_N denotes the projection that takes $\{x_0 \dots x_N\}$ to x_N .

Definition 1. A linear function $\widehat{(\ell, x_N)}$ is called a minimax a-posteriori estimation if the following condition holds:

$$\inf_{\{\tilde{x}_k\} \in \mathcal{G}_y^N} \sup_{\{x_k\} \in \mathcal{G}_y^N} |(\ell, x_N) - (\ell, \tilde{x}_N)| = \sup_{\{x_k\} \in \mathcal{G}_y^N} |(\ell, x_N) - \widehat{(\ell, x_N)}|$$

The non-negative number

$$\hat{\sigma}(\ell, N) = \sup_{\{x_k\} \in \mathcal{G}_y^N} |(\ell, x_N) - \widehat{(\ell, x_N)}|$$

is called a minimax a-posteriori error in the direction ℓ . A map

$$N \mapsto I_N = \dim\{\ell \in \mathbb{R}^n : \hat{\sigma}(\ell, N) < +\infty\}$$

is called an index of causality for the pair of systems (1)-(3).

Denote by $k \mapsto Q_k$ a recursive map that takes each $k \in \mathbb{N}$ to the matrix Q_k , where

$$(7) \quad \begin{aligned} Q_k &= H'_k R_k H_k + F'_k [S_{k-1} - S_{k-1} C_{k-1} W_{k-1}^+ C'_{k-1} S_{k-1}] F_k, \\ Q_0 &= F'_0 S F_0 + H'_0 R_0 H_0, W_k = Q_k + C'_k S_k C_k \end{aligned}$$

Let $k \mapsto r_k$ be a recursive map that takes each natural number k to the vector $r_k \in \mathbb{R}^n$, where

$$(8) \quad \begin{aligned} r_k &= F'_k S_{k-1} C_{k-1} W_{k-1}^+ r_{k-1} + H'_k R_k y_k, \\ r_0 &= H'_0 R_0 y_0 \end{aligned}$$

and to each number $i \in \mathbb{N}$ assign the number α_i , where

$$(9) \quad \begin{aligned} \alpha_i &= \alpha_{i-1} + (R_i y_i, y_i) - (W_{i-1}^+ r_{i-1}, r_{i-1}), \\ \alpha_0 &= (S g, g) + (R_0 y_0, y_0) \end{aligned}$$

The main result of this paper is formulated in the next theorem.

Theorem 1 (minimax recursive estimation). Suppose we are given a natural number N and a vector $\ell \in \mathbb{R}^n$. Then a necessary and sufficient condition for a minimax a-posteriori error $\hat{\sigma}(\ell, N)$ to be finite is that

$$(10) \quad Q_N^+ Q_N \ell = \ell$$

Under this condition we have

$$(11) \quad \hat{\sigma}(\ell, N) = [1 - \alpha_N + (Q_N^+ r_N, r_N)]^{\frac{1}{2}} (Q_N^+ \ell, \ell)^{\frac{1}{2}}$$

and

$$(12) \quad \widehat{(\ell, x_N)} = (\ell, Q_N^+ r_N)$$

Corrolary 1. The index of causality I_N for the pair of systems (1)-(3) can be represented as $I_N = \text{rank}(Q_N)$.

Corrolary 2 (minimax observer). *The online minimax observer is given by $k \mapsto \hat{x}_k = Q_k^+ r_k$ and*³

$$(13) \quad \hat{\rho}(N) = \min_{\{x_k\} \in \mathcal{G}_y^N} \max_{\{\tilde{x}_k\} \in \mathcal{G}_y^N} \|x_N - \tilde{x}_N\|^2 = \frac{[1 - \alpha_N + (Q_N \hat{x}_N, \hat{x}_N)]}{\min_i \{\lambda_i(N)\}}$$

where $\lambda_i(N)$ are eigen values of Q_N . In this case all possible realisations of (1) state vector x_N fill the ellipsoid $P_N(\mathcal{G}_y^N) \subset \mathbb{R}^n$, where

$$(14) \quad P_N(\mathcal{G}_y^N) = \{x : (Q_N x, x) - 2(Q_N \hat{x}_N, x) + \alpha_N \leq 1\}$$

Remark 1. If $\lambda_{\min}(H'_k R_k H_k)$ grows for $k = i, i+1, \dots$ then the minimax estimation error $\hat{\rho}(k)$ becomes smaller causing \hat{x}_k to get closer to the real state vector x_k .

In [7] Kalman's filtering problem for descriptor systems was investigated from the deterministic point of view. Authors recover Kalman's recursion to the time-variant descriptor system by a deterministic least square fitting problem over the entire trajectory: find a sequence $\{\hat{x}_{0|k}, \dots, \hat{x}_{k|k}\}$ that minimises the following fitting error cost

$$J_k(\{x_{i|k}\}_0^k) = \|F_0 x_{0|k} - g\|^2 + \|y_0 - H_0 x_{0|k}\|^2 + \sum_{i=1}^k \|F_i x_{i|k} - C_{i-1} x_{i-1|k}\|^2 + \|y_i - H_i x_{i|k}\|^2$$

assuming that the rank $\text{rank}_{H_k}^{F_k} \equiv n$. According to [7, p.8] the successive optimal estimates $\{\hat{x}_{0|k}, \dots, \hat{x}_{k|k}\}$ resulting from the minimisation of J_k can be found from the recursive algorithm

$$(15) \quad \begin{aligned} \hat{x}_{k|k} &= P_{k|k} F'_k (E + C_{k-1} P_{k-1|k-1} C'_{k-1})^{-1} C_{k-1} \hat{x}_{k-1|k-1} \\ &\quad + P_{k|k} H'_k R_k y_k, \hat{x}_{0|0} = P_{0|0} (F'_0 q + H'_0 y_0), \\ P_{k|k} &= (F'_k (E + C_{k-1} P_{k-1|k-1} C'_{k-1})^{-1} F_k + H'_k H_k)^{-1}, \\ P_{0|0} &= (F'_0 F_0 + H'_0 H_0)^{-1} \end{aligned}$$

Corrolary 3 (Kalman's filter recursion). *Suppose the rank $\text{rank}_{H_k}^{F_k} \equiv n$, and let $k \mapsto r_k$ be a recursive map that takes each natural number k to the vector $r_k \in \mathbb{R}^n$, where*

$$(16) \quad \begin{aligned} r_k &= H'_k y_k + F'_k C_{k-1} (C'_{k-1} C_{k-1} + Q_{k-1})^+_{k-1} r_{k-1}, \\ r_0 &= F'_0 q + H'_0 y_0 \end{aligned}$$

Then $Q_k^+ r_k = \hat{x}_{k|k}$ for each $k \in \mathbb{N}$, where $\hat{x}_{k|k}$ is given by (15) and $I_k = n$.

3. EXAMPLE

Let us set $H_0 = \begin{bmatrix} \frac{6}{10} & \frac{96}{100} & 0 \\ 1000 & 2\frac{3}{10} & 0 \\ 1 & \frac{1}{10} & 0 \\ 0 & 0 & 0 \end{bmatrix}, F_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$

$$C_k \equiv \begin{bmatrix} \frac{1}{40} & \frac{1}{2} & 0 \\ \frac{1}{10} & \frac{1}{4} & \frac{3}{10} \end{bmatrix}, H_k \equiv \begin{bmatrix} k * \frac{6}{10} & k & 0 \\ 100k & \frac{k}{100} & 0 \\ 0 & 0.005 & 150k * q(k) \\ 0.05 & 10k & 0 \end{bmatrix},$$

³We assume here that $\frac{1}{0} = +\infty$.

where $q(k) = 1$ if k is odd and otherwise $q(k) = 0$. We derive the output x_k of (1) and y_k assuming f_k, g_k to be bounded vector-functions on the whole real axis. Also we set $R_k = \text{diag}\{\frac{1}{11(k+1)}, \frac{1}{22(k+1)}, \frac{1}{33(k+1)}, \frac{1}{44(k+1)}\}$, $S_k = \text{diag}\{\frac{1}{35(k+1)}, \frac{1}{70(k+1)}\}$, $S = \text{diag}\{\frac{1}{60}, \frac{1}{120}\}$.

We derive \hat{x}_k from (8) and $\hat{\sigma}(e_i, k)$ from (11), e_i - i-ort. Note that the rank $\frac{F_{2k+1}}{H_{2k+1}} < 3$ and $I_{2k} = 3$, $I_{2k+1} < 3$. Thus $\hat{x}_{3,2k+1} = 0$,

$$[1 - \alpha_{2k+1} + (Q_{2k+1}^+ r_{2k+1}, r_{2k+1})]^{\frac{1}{2}} (Q_{2k+1}^+ \ell, \ell)^{\frac{1}{2}} = 0$$

but $|x_{3,2k+1} - \hat{x}_{3,2k+1}| > 0$. The dynamics of $x_{i,k}$, $\hat{x}_{i,k}$, $|x_{i,k} - \hat{x}_{i,k}|$ and $\hat{\sigma}(e_i, k)$ is described by figures 1-2.

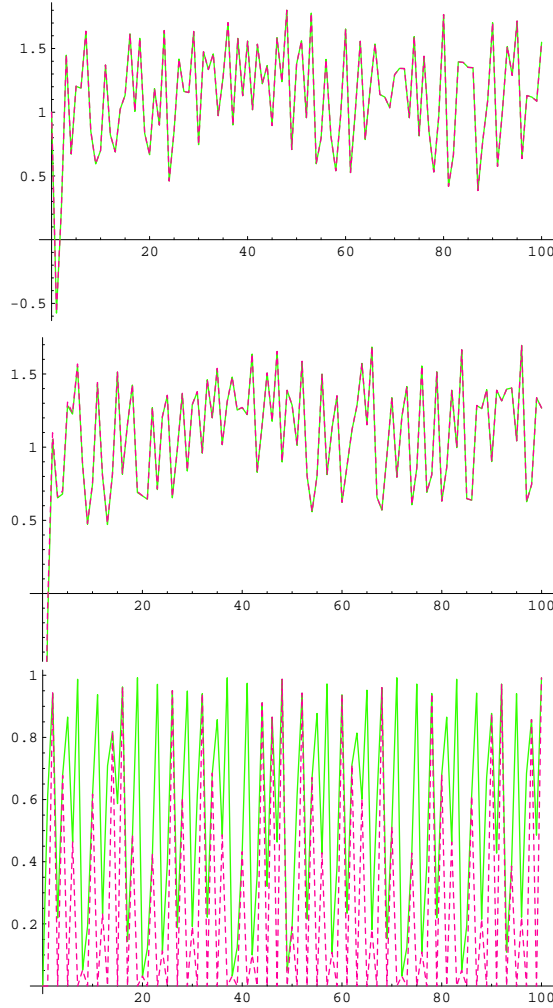


FIGURE 1. $N = 100$, output $x_{i,k}$ (solid) and observer $\hat{x}_{i,k}$ (dashed) to the left;

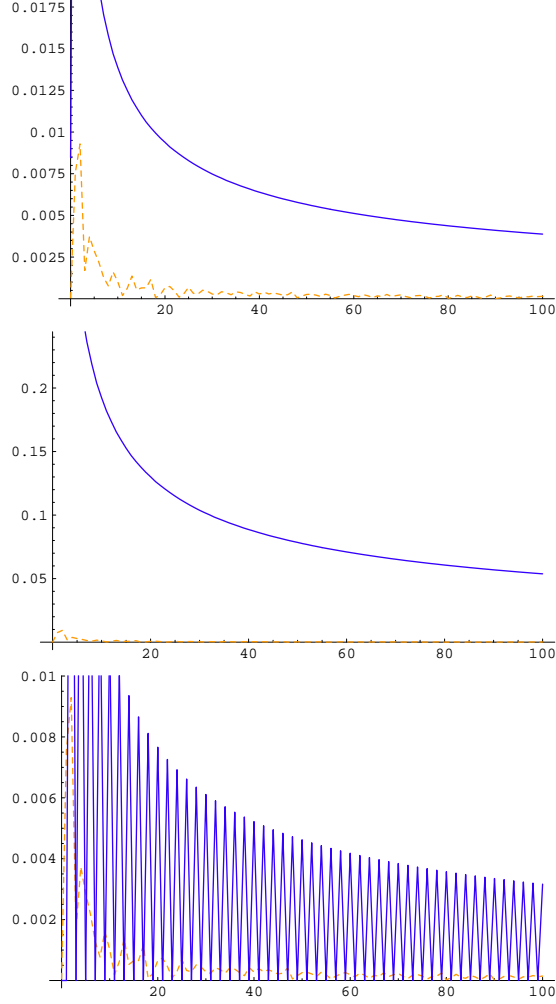


FIGURE 2. $N = 100$, real estimation error $|x_{i,k} - \hat{x}_{i,k}|$ (dashed) and minimax error $\hat{\sigma}(e_i, k)$ (solid) to the right.

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APPENDIX A. PROOFS.

Proof. Proof of Theorem 1. By definition, put

$$\mathbb{H} = \begin{pmatrix} H_0 & 0_{pn} & \dots & 0_{pn} \\ 0_{pn} & H_1 & \dots & 0_{pn} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{pn} & 0_{pn} & \dots & H_N \end{pmatrix}, \mathbb{F} = \begin{pmatrix} F_0 & 0_{mn} & 0_{mn} & \dots & 0_{mn} & 0_{mn} \\ -C_0 & F_1 & 0_{mn} & \dots & 0_{mn} & 0_{mn} \\ 0_{mn} & -C_1 & F_2 & \dots & 0_{mn} & 0_{mn} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{mn} & 0_{mn} & 0_{mn} & \dots & -C_{N-1} & F_N \end{pmatrix}$$

$$\mathcal{X} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \mathcal{Y} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix},$$

$$\mathcal{F} = \begin{bmatrix} q \\ f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{bmatrix}, \mathcal{G} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_N \end{bmatrix}.$$

By direct calculation we obtain $(\ell, x_N) = (\mathcal{L}, \mathcal{X})$,

$$\mathcal{G}_y^N = \{\mathcal{X} : \|\mathbb{F}\mathcal{X}\|_1^2 + \|\mathcal{Y} - \mathbb{H}\mathcal{X}\|_2^2 \leq 1\},$$

where $\|\mathcal{F}\|_1^2 = (Sq, q) + \sum_{k=0}^{N-1} (S_k f_k, f_k)$, $\|\cdot\|_2$ is induced by R_k on the same way. This implies

$$\sup_{\{x_k\} \in \mathcal{G}_y^N} |(\ell, x_N - \tilde{x}_N)| = \sup_{\mathcal{X} \in \mathcal{G}_y^N} |(\mathcal{L}, \mathcal{X}) - (\mathcal{L}, \tilde{\mathcal{X}})|$$

Denote by \mathbf{L} the set $R[\mathbb{F}' \ \mathbb{H}']$. We obviously get

$$\mathcal{L} \in \mathbf{L} \Leftrightarrow \sup_{\mathcal{X} \in \mathcal{G}_y^N} |(\mathcal{L}, \mathcal{X}) - (\mathcal{L}, \tilde{\mathcal{X}})| < +\infty$$

The application of Corollary 4 yields (10). Consider a vector $\mathcal{L} \in \mathbf{L}$. Clearly

$$\inf_{\mathcal{X} \in \mathcal{G}_y^N} (\mathcal{L}, \mathcal{X}) \leq (\mathcal{L}, \mathcal{X}) \leq \sup_{\mathcal{X} \in \mathcal{G}_y^N} (\mathcal{L}, \mathcal{X}), \mathcal{X} \in \mathcal{G}_y^N$$

Let c denotes $\frac{1}{2}(\sup_{\mathcal{X} \in \mathcal{G}_y^N} (\mathcal{L}, \mathcal{X}) + \inf_{\mathcal{X} \in \mathcal{G}_y^N} (\mathcal{L}, \mathcal{X}))$. Therefore

$$\begin{aligned} \sup_{\mathcal{X} \in \mathcal{G}_y^N} |(\mathcal{L}, \mathcal{X}) - (\mathcal{L}, \tilde{\mathcal{X}})| &= \\ \frac{1}{2}(s(\mathcal{L}|\mathcal{G}_y^N) + s(-\mathcal{L}|\mathcal{G}_y^N)) &+ |c - (\mathcal{L}, \tilde{\mathcal{X}})| \end{aligned}$$

hence

$$\begin{aligned} \hat{\sigma}(\ell, N) &= \frac{1}{2}(s(\mathcal{L}|\mathcal{G}_y^N) + s(-\mathcal{L}|\mathcal{G}_y^N)), \\ (\widehat{(\ell, x_N)}) &= \frac{1}{2}(s(\mathcal{L}|\mathcal{G}_y^N) - s(-\mathcal{L}|\mathcal{G}_y^N)), \end{aligned} \quad (17)$$

where $s(\cdot|\mathcal{G}_y^N)$ denotes the support function of \mathcal{G}_y^N . Clearly, \mathcal{G}_y^N is a convex closed set. Hence the equality $(\mathcal{L}, \tilde{\mathcal{X}}) = \widehat{(\ell, x_N)}$ is held for some $\tilde{\mathcal{X}} \in \mathcal{G}_y^N$. Thus, to conclude the proof we have to calculate $s(\mathcal{L}, \mathcal{G}_y^N)$. Let

$$\mathcal{G}_0^N = \{\mathcal{X} : \|\mathbb{F}\mathcal{X}\|^2 + \|\mathbb{H}\mathcal{X}\|^2 \leq \beta_N\}, \quad (18)$$

where $\beta_N = 1 - \alpha_N + (Q_N^+ r_N, r_N) \geq 0$.

Lema 1.

$$s(\mathcal{L}, \mathcal{G}_y^N) = (\ell, Q_N^+ r_N) + s(\mathcal{L}|\mathcal{G}_0^N) \quad (19)$$

It follows from the definition of \mathcal{G}_0^N that $s(\mathcal{L}|\mathcal{G}_0^N) = s(-\mathcal{L}|\mathcal{G}_0^N)$ hence (17) implies

$$\widehat{(\ell, x_N)} = (\ell, Q_N^+ r_N), \hat{\sigma}(\ell) = s(\mathcal{L}|\mathcal{G}_0^N)$$

The application of Lemma 2 completes the proof.

Lema 2.

$$(20) \quad s(\mathcal{L}|\mathcal{G}_0^N) = \begin{cases} \sqrt{\beta_N}(Q_N^+\ell, \ell)^{\frac{1}{2}}, [E - Q_N^+Q_N]\ell = 0, \\ +\infty, [E - Q_N^+Q_N]\ell \neq 0 \end{cases}$$

□

Let r_k denote \mathbb{R}^n -valued recursive map

$$(21) \quad \begin{aligned} r_k &= F'_k(S_{k-1} - S_{k-1}C_{k-1}P_{k-1}^+C'_{k-1}S_{k-1})f_{k-1} + \\ &F'_kS_{k-1}C_{k-1}W_{k-1}^+r_{k-1} + H'_kR_ky_k, \\ r_0 &= F'_0Sq + H'_0R_0y_0, P_k = C'_kS_kC_k + Q_k \end{aligned}$$

and set

$$\begin{aligned} J(\{x_k\}) &= \|F_0x_0 - g\|_S^2 + \|y_0 - H_0x_0\|_0^2 + \\ &\sum_{k=1}^N \|F_kx_k - C_{k-1}x_{k-1} - f_{k-1}\|_{k-1}^2 + \|y_k - H_kx_k\|_k^2 \end{aligned}$$

where $\|g\|_S^2 = (Sg, g)$, $\|f_k\|_k^2 = (S_kf_k, f_k)$, $\|y_i\|_i^2 = (R_iy_i, y_i)$.

Lema 3. Let $x \mapsto \hat{x}_k$ be a recursive map that takes any $k \in \mathbb{N}$ to $\hat{x}_k \in \mathbb{R}^n$, where

$$(22) \quad \begin{aligned} \hat{x}_k &= P_k^+(C'_kS_k(F_{k+1}\hat{x}_{k+1} - f_k) + r_k), \\ \hat{x}_N &= Q_N^+r_N, \end{aligned}$$

Then

$$\min_{\{x_k\}} J(\{x_k\}) = J(\{\hat{x}_k\})$$

Proof. By definition put $\Phi(x_0) := \|F_0x_0 - g\|_S^2 + \|y_0 - H_0x_0\|_0^2$

$$\Phi_i(x_i, x_{i+1}) := \|F_{i+1}x_{i+1} - C_ix_i - f_i\|_i^2 + \|y_{i+1} - H_{i+1}x_{i+1}\|_{i+1}^2$$

Then we obviously get

$$(23) \quad J(\{x_k\}) = \Phi(x_0) + \sum_{i=0}^{N-1} \Phi_i(x_i, x_{i+1})$$

Let us apply a modification of Bellman's method⁴ to the nonlinear programming task

$$J(\{x_k\}) \rightarrow \min_{\{x_k\}}$$

By definition put

$$\ell_1(x_1) := \min_{x_0} \{\Phi(x_0) + \Phi_0(x_0, x_1)\}$$

Using (7) and (21) one can get

$$\Phi(x_0) = (Q_0x_0, x_0) - 2(r_0, x_0) + \alpha_0 \geq 0, \alpha_0 := \|g\|_S^2 + \|y_0\|_0^2$$

On the other hand it's clear that

$$\ell_1(x_1) = \Phi(\hat{x}_0) + \Phi_0(\hat{x}_0, x_1) = (Q_1x_1, x_1) - 2(r_1, x_1) + \alpha_1 \geq 0,$$

where $\hat{x}_0 = P_0^+(r_0 + C'_0S_0(F_1x_1 - f_0))$

$$\alpha_1 := \alpha_0 + \|y_1\|_1^2 + \|f_0\|_0^2 - (P_0^+(r_0 - C'_0S_0f_0), r_0 - C'_0S_0f_0)$$

⁴So-called "Kievskiy venyk" method

Considering $\ell_1(x_1)$ as an induction base and assuming that

$$\begin{aligned}\ell_{i-1}(x_{i-1}) &= \min_{x_{i-2}} \{\Phi_{i-2}(x_{i-2}, x_{i-1}) + \ell_{i-2}(x_{i-2})\} = \\ &= (Q_{i-1}x_{i-1}, x_{i-1}) - 2(r_{i-1}, x_{i-1}) + \alpha_{i-1}\end{aligned}$$

now we are going to prove that

$$\begin{aligned}(24) \quad \ell_i(x_i) &= \min_{x_{i-1}} \{\Phi_{i-1}(x_{i-1}, x_i) + \ell_{i-1}(x_{i-1})\} = \\ &= (Q_i x_i, x_i) - 2(r_i, x_i) + \alpha_i\end{aligned}$$

Note that [10] for any convex function $(x, y) \mapsto f(x, y)$

$$y \mapsto \min\{f(x, y) | (x, y) : P(x, y) = y\}, P(a, b) = b$$

is convex. Thus taking into account the definition of $\ell_1(x_1)$ one can prove by induction that ℓ_{i-1} is convex and

$$\Phi_{i-1}(x_{i-1}, x_i) + \ell_{i-1}(x_{i-1}) \geq 0$$

Hence⁵ $Q_{i-1} \geq 0$, the set of global minimums Ψ_{i-1} of the quadratic function

$$x_{i-1} \mapsto \Phi_{i-1}(x_{i-1}, x_i) + (Q_{i-1}x_{i-1}, x_{i-1}) - 2(r_{i-1}, x_{i-1}) + \alpha_{i-1}$$

is non-empty and $\hat{x}_{i-1} \in \Psi_i$, where⁶

$$\hat{x}_{i-1} = (Q_{i-1} + C'_{i-1}S_{i-1}C_{i-1})^+(C'_{i-1}S_{i-1}(F_i x_i - f_{i-1}) + r_{i-1})$$

This implies

$$\begin{aligned}\ell_i(x_i) &= \Phi_{i-1}(\hat{x}_{i-1}, x_i) + \ell_{i-1}(\hat{x}_{i-1}) = \\ &= (Q_i x_i, x_i) - 2(r_i, x_i) + \alpha_i,\end{aligned}$$

where

$$\begin{aligned}\alpha_i &= \alpha_{i-1} + (R_i y_i, y_i) + (S_{i-1} f_{i-1}, f_{i-1}) - \\ &= (P_{i-1}^+(r_{i-1} - C'_{i-1}S_{i-1}f_{i-1}), r_{i-1} - C'_{i-1}S_{i-1}f_{i-1}),\end{aligned}$$

Therefore, we obtain

$$\min_{x_N} \ell_N(x_N) = \ell_N(\hat{x}_N) = \alpha_N - (r_N, Q_N^+ r_N), \hat{x}_N = Q_N^+ r_N$$

so that $\min_{\{x_k\}} J(\{x_k\}) = J(\{\hat{x}_k\})$. □

Corrolary 4. Suppose $\mathcal{L} = [0 \dots \ell]$; then

$$\mathcal{L} \in \mathcal{R}[\mathbb{F}' \quad \mathbb{H}'] \Leftrightarrow [E - Q_N^+ Q_N] \ell = 0$$

and

$$\|[\mathbb{F}' \quad \mathbb{H}']^+ \mathcal{L}\|^2 = (Q_N^+ \ell, \ell)$$

Proof. Suppose $S_k = E, R_k = E$ for a simplicity. If $\mathcal{L} \in \mathcal{R}[\mathbb{F}' \quad \mathbb{H}']$ then

$$F'_N z_N + H'_N u_N = \ell, \quad F'_k z_k + H'_k u_k - C'_k z_{k+1} = 0 \quad (*),$$

for some $z_k \in \mathbb{R}^m, u_k \in \mathbb{R}^p$. Let's find the projection $\{(\hat{z}_k, \hat{u}_k)\}_{k=0}^N$ of the vector $\{(z_k, u_k)\}_{k=0}^N$ onto the range of the matrix $\begin{bmatrix} \mathbb{F} \\ \mathbb{H} \end{bmatrix}$. Lemma 3 implies

$$\hat{z}_0 = F_0 \hat{x}_0, \hat{z}_k = F_k \hat{x}_k - C_{k-1} \hat{x}_{k-1}, \hat{u}_k = H_k \hat{x}_k, \quad (**)$$

⁵The function $x \mapsto (Ax, x) - 2(x, q) + c$ is convex iff $A = A' \geq 0$.

⁶The vector \hat{x}_{i-1} has the smallest norm among other points of theminimum.

where

$$\begin{aligned}\hat{x}_k &= P_k^+(C'_k F_{k+1} \hat{x}_{k+1} + r_k - C'_k z_{k+1}), \hat{x}_N = Q_N^+ r_N, \\ r_k &= F'_k C_{k-1} P_{k-1}^+ r_{k-1} + F'_k (E - C_{k-1} P_{k-1}^+ C'_{k-1}) z_k + \\ &+ H'_k u_k, r_0 = F'_0 z_0 + H'_0 u_0, P_k = C'_k C_k + Q_k\end{aligned}$$

(*) implies $r_k = C'_k z_{k+1}, k = 0, \dots, N-1, r_N = \ell$ thus $\hat{x}_N = Q_N^+ \ell, \hat{x}_k = P_k^+ C'_k F_{k+1} \hat{x}_{k+1}$ or $\hat{x}_k = \Phi(k, N) Q_N^+ \ell$,

$$\Phi(k, N) = P_k^+ C'_k F_{k+1} \Phi(k+1, N), \Phi(s, s) = E$$

Combining this with (**) we obtain

$$\begin{aligned}(25) \quad \hat{z}_k &= (F_k \Phi(k, N) - C_{k-1} \Phi(k-1, N)) Q_N^+ \ell, \\ \hat{u}_k &= H_k \Phi(k, N) Q_N^+ \ell, \hat{z}_0 = F_0 \Phi(0, N) Q_N^+ \ell\end{aligned}$$

By definition, put $U(0) = Q_0$,

$$\begin{aligned}U(k) &= \Phi'(k-1, k) U(k-1) \Phi(k-1, k) + \\ &H'_k H_k + F_k (E - C_{k-1} P_{k-1}^+ C'_{k-1})^2 F_k\end{aligned}$$

It now follows that

$$\|[\mathbb{F}' \quad \mathbb{H}']^+ \mathcal{L}\|^2 = \sum_0^N \|\hat{z}_N\|^2 + \|\hat{u}_N\|^2 = (U(N) Q_N^+ \ell, Q_N^+ \ell)$$

It's easy to prove by induction that $Q_k = U(k)$.

Since

$$\mathcal{L} \in R[\mathbb{F}' \quad \mathbb{H}']$$

we obtain by substituting \hat{z}_k, \hat{u}_k into (*)

$$F'_N \hat{z}_N + H'_N \hat{u}_N = \ell$$

On the other hand (7) and (25) imply

$$F'_N \hat{z}_N + H'_N \hat{u}_N = \ell \Rightarrow [E - Q_N^+ Q_N] \ell = 0$$

Suppose that $[E - Q_N^+ Q_N] \ell = 0$. To conclude the proof we have to show that

$$(\ell, x_N) = (Q_N^+ \ell, Q_N x_N) = 0, \forall [x_0 \dots x_N] \in \mathcal{N}[\mathbb{F}, \mathbb{H}]$$

By induction, fix $N = 0$. If $F_0 x_0 = 0, H_0 x_0 = 0$, then $Q_0 x_0 = 0$. We say that $[x_0 \dots x_k] \in \mathcal{N}[\mathbb{F}, \mathbb{H}]$ if

$$F_0 x_0 = 0, H_0 x_0 = 0, F_s x_s = C_{s-1} x_{s-1}, H_s x_s = 0,$$

Suppose $Q_{k-1} x_{k-1} = 0, \forall [x_0 \dots x_{k-1}] \in \mathcal{N}[\mathbb{F}, \mathbb{H}]$ and fix any $[x_0 \dots x_k] \in \mathcal{N}[\mathbb{F}, \mathbb{H}]$. Then $F_k x_k = C_{k-1} x_{k-1}, H_k x_k = 0$. Combining this with (7) we obtain

$$Q_k x_k = F'_k (E - C_{k-1} P_{k-1}^+ C'_{k-1}) C_{k-1} x_{k-1} \quad (*)$$

We show that $Q_k \geq 0$ in the proof of Theorem 1. One can see that

$$\begin{aligned}&\left[\begin{array}{c} C_{k-1} \\ Q_{k-1}^{\frac{1}{2}} \end{array} \right]^+ = \\ &[(C'_{k-1} C_{k-1} + Q_{k-1})^+ C'_{k-1}, (C'_{k-1} C_{k-1} + Q_{k-1})^+ Q_{k-1}^{\frac{1}{2}}]\end{aligned}$$

Since

$$\left[\begin{array}{c} C_{k-1} \\ Q_{k-1}^{\frac{1}{2}} \end{array} \right] \left[\begin{array}{c} C_{k-1} \\ Q_{k-1}^{\frac{1}{2}} \end{array} \right]^+ \left[\begin{array}{c} C_{k-1} \\ Q_{k-1}^{\frac{1}{2}} \end{array} \right] x_{k-1} = \left[\begin{array}{c} C_{k-1} \\ Q_{k-1}^{\frac{1}{2}} \end{array} \right] x_{k-1}$$

we obviously get

$$C_{k-1}P_{k-1}^+C_{k-1}'C_{k-1}x_{k-1} = C_{k-1}x_{k-1} \Rightarrow Q_kx_k = 0$$

as it follows from (*). This completes the proof. \square

Proof. Proof of Lemma 1. Taking into account the definitions of the matrices \mathbb{F}, \mathbb{H} and (6) we clearly have

$$\mathcal{G}_y^N = \{\mathcal{X} : \|\mathbb{F}\mathcal{X}\|^2 + \|\mathcal{Y} - \mathbb{H}\mathcal{X}\|^2 \leq 1\}$$

Let $\hat{\mathcal{X}}$ be a minimum of the quadratic function $\mathcal{X} \mapsto \|\mathbb{F}\mathcal{X}\|^2 + \|\mathcal{Y} - \mathbb{H}\mathcal{X}\|^2$. It now follows that

$$\mathcal{G}_y^N = \hat{\mathcal{X}} + \mathcal{G}_0^N \Rightarrow s(\mathcal{L}|\mathcal{G}_0^N) = (\mathcal{L}, \hat{\mathcal{X}}) + s(\mathcal{L}|\mathcal{G}_0^N)$$

The application of Lemma 3 yields

$$(\mathcal{L}, \hat{\mathcal{X}}) = (\ell, Q_N^+r_N)$$

This completes the proof. \square

Proof. Proof of Lemma 2. Suppose the function $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$ is convex and closed. Then [10] the support function $s(\cdot|\{x : f(x) \leq 0\})$ of the set $\{x : f(x) \leq 0\}$ is given by

$$s(z|\{x : f(x) \leq 0\}) = \text{cl} \inf_{\lambda \geq 0} \{\lambda f^*(\frac{z}{\lambda})\}$$

To conclude the proof it remains to compute the support function of \mathcal{G}_0^N according to this rule and then apply Corollary 4. \square

Proof. Proof of Corollary 3. The proof is by induction on k . For $k = 0$, there is nothing to prove. The induction hypothesis is $P_{k-1|k-1} = Q_{k-1}^{-1}$. Suppose S is $n \times n$ -matrix such that $S = S' > 0$, A is $m \times n$ -matrix; then

$$(26) \quad A(S^{-1} + A'A)^{-1} = (E + ASA')^{-1}AS$$

Using (26) we get

$$(27) \quad ASA' = [E + ASA']A[A'A + S^{-1}]^{-1}A'$$

Combining (27) with the induction assumption we get the following

$$\begin{aligned} E + C_{k-1}P_{k-1|k-1}C_{k-1}' &= \\ E + [E + C_{k-1}P_{k-1|k-1}C_{k-1}'] \times \\ &\times C_{k-1}[Q_{k-1} + C_{k-1}'C_{k-1}]^{-1}C_{k-1}' \end{aligned}$$

By simple calculation from the previous equality follows

$$\begin{aligned} E - C_{k-1}(Q_{k-1} + C_{k-1}'C_{k-1})^{-1}C_{k-1}' &= \\ (E + C_{k-1}P_{k-1|k-1}C_{k-1}')^{-1} & \end{aligned}$$

Using this and (7),(15) we obviously get $Q_k^{-1} = P_{k|k}$.

It follows from the definitions that $Q_0^{-1}r_0 = \hat{x}_{0|0}$. Suppose that $Q_{k-1}^{-1}r_{k-1} = \hat{x}_{k-1|k-1}$. The induction hypothesis and (26) imply

$$\begin{aligned} (E + C_{k-1}P_{k-1|k-1}C_{k-1}')^{-1}C_{k-1}\hat{x}_{k-1|k-1} &= \\ C_{k-1}(C_{k-1}'C_{k-1} + Q_{k-1})_{k-1}^{-1}r_{k-1} & \end{aligned}$$

Combining this with (15), (16) and using $Q_k^{-1} = P_{k|k}$ we obtain

$$\hat{x}_{k|k} = Q_k^{-1}(F'_k C_{k-1}(C'_{k-1} C_{k-1} + Q_{k-1})_{k-1}^+ r_{k-1} + H'_k y_k)$$

This concludes the proof. \square

Proof. Proof of Corollary 2. If $I_k < n$ then $\text{rank}(Q) < n$ hence $\lambda_{\min}(Q_k) = 0$. In this case there is a direction $\ell \in \mathbb{R}^n$ such that $\hat{\sigma}(\ell, k) = +\infty$. So $\hat{\rho}(k) = +\infty$. If $I_k = n$ then we clearly have

$$\begin{aligned} & \min_{\{x_k\} \in \mathcal{G}_y^N} \max_{\{\tilde{x}_k\} \in \mathcal{G}_y^N} \|x_N - \tilde{x}_N\|^2 = \\ & \min_{\{x_k\} \in \mathcal{G}_y^N} \max_{\{\tilde{x}_k\} \in \mathcal{G}_y^N} \left\{ \max_{\|\ell\|=1} |(\ell, x_N - \tilde{x}_N)| \right\}^2 = \\ & \left\{ \min_{\mathcal{G}_y^N} \max_{\|\ell\|=1} \max_{\{\tilde{x}_k\} \in \mathcal{G}_y^N} |(\ell, x_N - \tilde{x}_N)| \right\}^2 \geq \\ & \left\{ \max_{\|\ell\|=1} \min_{\{x_k\} \in \mathcal{G}_y^N} \max_{\{\tilde{x}_k\} \in \mathcal{G}_y^N} |(\ell, x_N - \tilde{x}_N)| \right\}^2 = \\ & [1 - \alpha_N + (Q_N^+ r_N, r_N)] \max_{\|\ell\|=1} (Q_N^+ \ell, \ell) = \\ & \frac{[1 - \alpha_N + (Q_N^+ r_N, r_N)]}{\min_i \{\lambda_i(N)\}} \end{aligned}$$

On the other hand Theorem 1 implies

$$\begin{aligned} & \max_{\{\tilde{x}_k\} \in \mathcal{G}_y^N} \|\hat{x}_N - \tilde{x}_N\|^2 = \\ & \left\{ \max_{\|\ell\|=1} \max_{\{\tilde{x}_k\} \in \mathcal{G}_y^N} |(\ell, x_N - \tilde{x}_N)| \right\}^2 = \\ & \left\{ \max_{\|\ell\|=1} [1 - \alpha_N + (Q_N^+ r_N, r_N)]^{\frac{1}{2}} (Q_N^+ \ell, \ell)^{\frac{1}{2}} \right\}^2 \end{aligned}$$

It follows now from $I_N = n$ that \mathcal{G}_y^N is a bounded set.

The equality $I_N = n$ implies $[E - Q_N^+ Q_N] = 0$ for a given N . It follows from Lemmas 1,2 that

$$(28) \quad \begin{aligned} & s(\ell | P_N(\mathcal{G}_y^N)) = s(P'_N \ell | \mathcal{G}_y^N) = s(\mathcal{L} | \mathcal{G}_y^N) = \\ & (\ell, Q_N^+ r_N) + \sqrt{\beta_N} (Q_N^+ \ell, \ell)^{\frac{1}{2}} \end{aligned}$$

for any $\ell \in \mathbb{R}^n$. By Young's theorem [10], (28), so that

$$\begin{aligned} & P_N(\mathcal{G}_y^N) = \{x \in \mathbb{R}^n : (x, \ell) \leq s(\ell | P_N(\mathcal{G}_y^N)), \forall \ell \in \mathbb{R}^n\} = \\ & \{x \in \mathbb{R}^n : \sup_{\ell} \{(x, \ell) - (\ell, \hat{x}_N) - \sqrt{\beta_N} (Q_N^+ \ell, \ell)^{\frac{1}{2}}\} \leq 0\} = \\ & \{x \in \mathbb{R}^n : (Q_N x, x) - 2(Q_N \hat{x}_N, x) + \alpha_N \leq 1\} \end{aligned}$$

\square

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